PROPAGATION AND REFLECTION OF ULTRASONIC BEAMS IN CRYSTALS

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A simple derivation is given to arrive at equations for finding the directions of sound beams in crystals; reflection from the free face is considered. The possibilities of using the shadow method of visualizing acoustic fields and multiply reflected pulse techniques are demonstrated as methods for the experimental observation of beams in crystals. Numerical calculations of the beam directions in quartz yield excellent agreement with experiment.

As is generally known, when electromagnetic waves propagate in a crystal the direction of energy transfer (beam) does not in general coincide with the wave vector. A similar effect is seen in the case of ultrasonic waves propagating in an elastically anisotropic medium.

The most general approach to determining the directions of ultrasonic beams is basically to determine the group velocity [1]. The group velocity vector specifies the direction of energy transfer and is equal to

$$\mathbf{v}_b = \partial\omega / \partial\mathbf{q}, \qquad (1)$$

where ω is the frequency and $\mathbf{q}$ is the wave vector.

The absolute value of $\mathbf{q}$ in the crystal depends on the directions of propagation and polarization of the elastic wave. Writing the equation for the wave vector surface in the form $F(q_i, \omega) = 0$ and considering ω to be a function of q_i, we have

$$\frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial \omega} \cdot \frac{\partial \omega}{\partial q_i} = 0.$$

Then the components of the beam velocity vector are

$$v_{b_j} = -\frac{\partial F}{\partial q_i} \Big/ \frac{\partial F}{\partial \omega}. \qquad (2)$$

Since the vector $\partial F / \partial \mathbf{q}$ is perpendicular to the surface $F = 0$, it is evident that the direction of the sound beam will be given normal to the corresponding point of the wave vector surface.

For our later computations, we will need to ascertain the form of $F(q_i, \omega)$. Consider the general case of ultrasonic propagation in a piezoelectric crystal (with restriction to the linear piezoeffect). The desired equations relating the elastic and electric variables are

$$\left.\begin{aligned} \rho \cdot \ddot{U}_i &= \frac{\partial \sigma_{ik}}{\partial x_k} = c^E_{iklm} \cdot \frac{\partial u_{lm}}{\partial x_k} - e_{j,ik} \cdot \frac{\partial E_j}{\partial x_k}, \\ D_p &= \varepsilon^u_{pq} \cdot E_q + 4\pi \cdot e_{p,rs} \cdot u_{rs}. \end{aligned}\right\} \qquad (3)$$

Here U_i represents the components of the displacement vector in the elastic wave, u_{lm} the components of the strain tensor, D_p the components of the induction vector, E_q the components of the electric field, c^E_{iklm} the elastic moduli for steady fields, $e_{j,ik}$ the piezoelectric constants, ϵ^u_{pq} the dielectric permeabilities at constant strain, ρ the crystal density.

These equations should be solved in conjunction with the additional provision that $\operatorname{div} \mathbf{D} = 0$ and $\operatorname{rot} \mathbf{E} = 0$. If we assume as the solution a plane monochromatic wave, then for the components of the displacement vector we obtain from (3) (see [1]) the set of equations

$$\left\{\rho \cdot \omega^2 \cdot \delta_{im} - c^E_{iklm} \cdot q_l \cdot q_k - \frac{4\pi (e_{j,ki} \cdot q_j \cdot q_k)(e_{p,rm} \cdot q_p \cdot q_r)}{\varepsilon^r_{pq} \cdot q_p \cdot q_q}\right\} U_m = 0. \qquad (4)$$

Here $q_l = q \cdot l_l$, l_l is the direction cosine of the wave vector. In the interest of space, we introduce the notation

$$\Gamma_{im} = c^E_{iklm} \cdot q_l q_k + \frac{4\pi (e_{j,ki} \cdot q_j \cdot q_k) \cdot (e_{p,rm} \cdot q_p \cdot q_r)}{\varepsilon^r_{pq} \cdot q_q \cdot q_p}. \qquad (5)$$

The system (4) has nonzero values provided

$$|\rho\cdot\omega^2\cdot\delta_{im} - \Gamma_{im}| = 0. \qquad (6)$$

After several algebraic manipulations, Eq. (6) becomes,

$$F(q_i, \omega) = \sum_{k=1}^{3} \frac{\beta_k}{\rho\omega^2 - \Gamma_{kk} + \beta_k} - 1 = 0, \quad (7)$$

where $\beta_1 = \frac{\Gamma_{12}\cdot\Gamma_{13}}{\Gamma_{23}}$, $\beta_2 = \frac{\Gamma_{12}\cdot\Gamma_{23}}{\Gamma_{13}}$, $\beta_3 = \frac{\Gamma_{13}\cdot\Gamma_{23}}{\Gamma_{12}}$.

Equations (6) and (7) represent two different ways of writing the wave vector surface equation. Given the direction of propagation, from these we can find three values of q corresponding in general to quasi-longitudinal and quasi-transverse waves.* Consequently, the set $\mathbf{q}$ forms three wave vector surfaces.

In practical problems it is often convenient to apply the concept of the normal velocity $v_n = \omega/q$, which determines the velocity of the wave front (this quantity is the one measured in determining the velocity of ultrasound). The equation of the normal velocity surface is obtained from (6) or (7) by substituting therein $\omega \to v_{(n)}$, $\Gamma_{im} \to Q_{im}$, $\beta_k \to \alpha_k$, subject to the following obvious relations:

$$\left.\begin{aligned} Q_{im} &= \frac{1}{q^2}\cdot\Gamma_{im} \\ \alpha_k &= \frac{1}{q^2}\cdot\beta_k \end{aligned}\right\}. \qquad (8)$$

The normal velocity and wave vector surfaces are mutually inverse and are readily transformed one into the other.

As is evident from Eq. (5), the velocity of sound in the crystal depends on the presence of a piezoelectric correction. Thus, for example, in quartz the modulus is equal to $c^E_{1111} + \frac{4\pi\cdot e^2_{111}}{\varepsilon^u_{II}} = c^D_{1111}$ for the case of longitudinal waves propagating along the X axis, whereas transverse waves propagating along the Z axis are governed by c^E_{2332} (since $e_{332} = e_{331} = 0$). In the first case the field of piezoelectric reaction is directed along the wave, in the second it is perpendicular to it; the effective elastic moduli in both instances therefore correspond to different electric conditions, either constant induction or constant field.

The direction cosines p_k of the displacement vectors of each wave are found from Eq. (4) with the substitution $U_m = p_m\cdot U$:

$$(\rho\cdot\omega^2\cdot\delta_{im} - \Gamma_{im})\cdot p_m = 0.$$

The solution of this equation can be written in the form

$$p_k = \frac{A\cdot\beta_k^{\frac{1}{2}}}{\rho\omega^2 - \Gamma_{kk} + \beta_k}, \qquad (9)$$

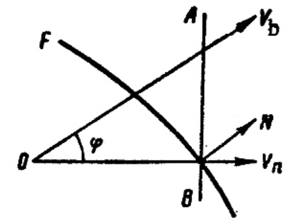

Fig. 1.

where A follows from the condition $\sum_{k=1}^{3} p_k^2 = 1$.

The displacement vectors of all three waves are mutually orthogonal.

On the basis of the above relations we readily obtain the final expressions for determining the directions of sound beams in a crystal. Making use of Eq. (7), we find

$$\frac{\partial F}{\partial q_i} = \sum_{k=1}^{3} \frac{\frac{\partial\beta_k}{\partial q_i}\cdot(\rho\omega^2 - \Gamma_{kk}) + \beta_k\cdot\frac{\partial\Gamma_{kk}}{\partial q_i}}{(\rho\omega^2 - \Gamma_{kk} + \beta_k)^2},$$

$$\frac{\partial F}{\partial\omega} = -\sum_{k=1}^{3} \frac{2\rho\cdot\omega\cdot\beta_k}{(\rho\omega^2 - \Gamma_{kk} + \beta_k)^2}.$$

For practical calculations, it is convenient to use, instead of q_i, the direction cosines of the wave vector. After relatively straightforward transformations of the above equations, taking (8) and (9) into account, we arrive at

$$\frac{\partial F}{\partial q_i} = \frac{q}{A^2}\cdot L_i, \quad \frac{\partial F}{\partial\omega} = -\frac{q}{A^2}\cdot 2\cdot\rho\cdot v_{(n)},$$

where

$$L_i = \sum_{k=1}^{3} p_k^2\cdot\left[\frac{1}{\alpha_k}\cdot\frac{\partial\alpha_k}{\partial l_i}\cdot(\rho\cdot v^2_{(n)} - Q_{kk}) + \frac{\partial Q_{kk}}{\partial l_i}\right]$$

$$= C_{ik} l_m P_k\cdot(P_l l_m + P_m\cdot l_i). \qquad (10)$$

The velocity of sound along the beam becomes

$$v_b = (v^2_{b_1} + v^2_{b_2} + v^2_{b_3})^{\frac{1}{2}} = \frac{1}{2\rho\cdot v_{(n)}}\left(\sum_{i=1}^{3} L_i^2\right)^{\frac{1}{2}} \qquad (11)$$

and the direction cosines of the beam vector are

$$\lambda_i = \frac{L_i}{\left(\sum_{i=1}^{3} L_i^2\right)^{\frac{1}{2}}}. \qquad (12)$$

* In crystals having different symmetry there are the so-called specific directions by which only "pure" waves are propagated (the direction of the displacement in each of the three waves is parallel or perpendicular to the wave vector) [2]. They can be calculated for every crystal and are determined by the stipulation that one of the displacement vectors is proportional to the wave vector.

On the basis of (10), making use of identity transformations of the type $\sum_{i=1}^{3} l_i \cdot \frac{\partial \alpha_k}{\partial l_i} = 2\alpha_k$, $\sum_{i=1}^{3} l_i \frac{\partial Q_{nm}}{\partial l_i} = 2 \cdot Q_{nm}$, we find

$$\sum_{i=1}^{3} L_i\, l_i = \sum_{k=1}^{3}\sum_{i=1}^{3} p_k^2 \cdot l_i \tag{13}$$

$$\times \left[\frac{1}{\alpha_k}\frac{\partial \alpha_k}{\partial l_i} \cdot (\rho v_{(n)})^2 - Q_{kk}) + \frac{\partial Q_{kk}}{\partial l_i}\right] = 2\rho \cdot v_{(n)}^2.$$

It follows from Eqs. (13) and (11) that

$$v_b = \frac{v_{(n)}}{\sum_{i=1}^{3} \lambda_i \cdot l_i} = \frac{v_{(n)}}{\cos \varphi}.$$

Figure 1 illustrates the directions of $\mathbf{v}_{(n)}$ and $\mathbf{v}_b$ with respect to the wave vector surface. At each instant the front of a wave propagating in the crystal will be tangent to the beam velocity surface. In other words, the beam surface itself will be the envelope of plane wave fronts. This rule, which is known from the optics of anisotropic media [3], can be used for a graphical determination of the beam directions based on sectional curves of the normal velocity surface in the elastic group symmetry planes of the crystal [4].

We note that the propagation of sound beams in crystals was investigated previously by Musgrave [5]. However, the method used in our paper to determine v_b and λ_i is far simpler, and the final expressions are obtained in a form suitable for practical calculations.

We will consider briefly the problem of reflection of sound waves propagating inside a crystal away from the face. As the reflecting surface we consider the plane $x_i = 0$ and assume that it is in contact with air, thus permitting use of the boundary conditions for a free surface as a very good approximation:

$$\sigma_{ik} \cdot n_i = 0. \tag{14}$$

Here $\sigma_{ik} = c_{iklm} \cdot \frac{1}{2}\left(\frac{\partial U_l}{\partial x_m} + \frac{\partial U_m}{\partial x_l}\right)$, the piezoelectric effect being neglected, and n_i is the projection of the outward normal on the axis X_i. Representing the incident and reflected wave in the form $U_l \cdot \exp i\,(\mathbf{qr} - \omega t)$, substituting this into (14) and dropping the time factor, we obtain

$$c_{iklm} \cdot n_i \cdot [(q_m^0 \cdot U_l^0 + q_l^0 \cdot U_m^0) \cdot e^{iq^0 \cdot r}$$

$$+ \sum_j (q_m^j \cdot U_l^j + q_l^j \cdot U_m^j) \cdot e^{iq^j \cdot r}] = 0. \tag{15}$$

The superscript 0 refers to the incident wave, j to the reflected waves.

The condition (15) must be satisfied identity-wise for any point of the plane $x_i = 0$, whence follows the equality of the tangential components of the wave vectors of the incident and each reflected wave:

$$q_k^0 = q_k^j \quad \text{for} \quad k \neq i. \tag{16}$$

The vector $\mathbf{q}^j$ lies in the plane of incidence. Using α to denote the angle between the normal at the point of incidence and the wave vector, we have $q^0 \sin \alpha^0 = q^j \sin \alpha^j$, which is equivalent to the relation

$$\frac{\sin \alpha^0}{v_{(n)}^0} = \frac{\sin \alpha^j}{v_{(n)}^j};$$

since $v_n^j = f(\alpha^j)$, the latter expression determines the direction of the reflected waves inexplicitly. Specifically, if the wave normal is perpendicular to the reflecting surface, the sound beam must be reflected in the direction of incidence (but the beam itself need not be perpendicular to the surface). It is necessarily pointed out in addition that in general the reflected beam does not lie in the plane of incidence unless the latter is simultaneously a symmetry plane of the crystal itself.

The set of Eqs. (15), after dropping the phase multipliers, gives the relation between the amplitudes of the incident and reflected waves, i.e., the values of the reflection coefficients. The condition (16) is conveniently used to determine the directions of the reflected waves graphically, on the basis of cross sections of the wave vectors.

We consider finally the possibilities of experimentally observing sound beams. For optically transparent crystals, quartz in particular, we applied the shadow method of visualizing sound fields (Toepler method). We will consider the theory of the method in application to crystals of any symmetry.

We will adopt as our coordinate system the principal axes of the crystal dielectric tensor ϵ_{ik}; it thus cannot coincide with the crystallographic axes. In the absence of sound the velocity of light along any line is determined by the Fresnel ellipse

$$\varepsilon_{ik}^{(0)} \cdot x_i \cdot x_k = 1. \tag{17}$$

When the crystal is strained ϵ_{ik} has the form [1]

$$\varepsilon_{ik} = \varepsilon_{ik}^{(0)} + a_{iklm} \cdot u_{lm}, \tag{18}$$

where a_{iklm} is the tensor optical elastic constant, which in general is asymmetric with respect to interchangeability of the subscript pairs ik and lm, i.e., $a_{iklm} \neq a_{lmik}$. The zero components of a_{iklm} correspond to the c_{iklm} of the given crystallographic system.

Under the influence of a sound wave the index of refraction (velocity of light) varies periodically. It is evident from (17) and (18) that this variation is connected both with rotation and with a change in length of the Fresnel ellipsoid. Let a beam of light propagate along one axis, say X. Straightforward computations give the angle of rotation about the axis,

$$\operatorname{tg} 2\theta = \frac{2a_{23lm}\cdot u_{lm}}{(\varepsilon_{22}^{(0)} + a_{22lm}\cdot u_{lm}) - (\varepsilon_{33}^{(0)} + a_{33lm}\cdot u_{lm})} \tag{19}$$

and the variations in effective values of ϵ_{ik},

$$\begin{aligned}\varepsilon_{22} &= \frac{1}{2}\Big\{(\varepsilon_{22}^{(0)} + a_{22lm}\cdot u_{lm}) + (\varepsilon_{33}^{(0)} + a_{33lm}\cdot u_{lm}) \\ &+ \sqrt{[(\varepsilon_{22}^{(0)} + a_{22lm}\cdot u_{lm}) - (\varepsilon_{33}^{(0)} + a_{33lm}\cdot u_{lm})]^2 + 4(a_{23lm}\cdot u_{lm})^2}\Big\}, \\ \varepsilon_{33} &= \frac{1}{2}\Big\{(\varepsilon_{22}^{(0)} + a_{22lm}\cdot u_{lm}) + (\varepsilon_{33}^{(0)} + a_{33lm}\cdot u_{lm}) \\ &- \sqrt{[(\varepsilon_{22}^{(0)} + a_{22lm}\cdot u_{lm}) - (a_{33}^{(0)} + a_{33lm}\cdot u_{lm})]^2 + 4(a_{23lm}\cdot u_{lm})^2}\Big\}.\end{aligned} \tag{20}$$

Completely analogous expressions can be written down for the cases that arise when the light beam propagates along the axes Y and Z.

Analysis of the expressions (19) and (20) reveals that longitudinal, quasi-longitudinal, and quasi-periodic waves always elicit a change in the velocity of light and should therefore be observable by the shadow method. Transverse waves frequently in certain instances cannot cause a change in the velocity of light, their ability to do so depending on the symmetry of the crystal and direction of propagation of the light beam (direction of observation). This is clarified in the example of quartz, whose principal dielectric axes of course correspond to the crystallographic axes. Let a transverse sound wave propagate along Z, the light beam along X. If the displacement vector of the sound wave coincides with axis Y, then $a_{2223} \neq 0$, $a_{2323} \neq 0$ so that, consequently, $\varepsilon_{22} \neq \varepsilon_{22}^{(0)}$, $\varepsilon_{33} \neq \varepsilon_{33}^{(0)}$. For a wave with a displacement along X, $a_{2213} = a_{3313} = a_{2313} = 0$ and $\varepsilon_{22} = \varepsilon_{22}^{(0)}$, $\varepsilon_{33} = \varepsilon_{33}^{(0)}$. Consequently, only the first wave will be observed. It can similarly be shown that both such waves can be observed when the light beam is directed along the axis Y. In this sense, the possibilities afforded by the shadow method are somewhat different for crystals than those for isotropic media, where it is impossible to observe a transverse wave with

Crystal. axis	Wave type	Velocity $v(n)\cdot 10^{-5}$ cm/sec	Direction of displ. vector	Beam direction	Beam deflection from normal
X	longitudinal	5.75	$p_1=1;\ p_2=p_3=0$	$\lambda_1=1;\ \lambda_2=\lambda_3=0$	0°
	transverse	5.10	$p_1=0;\ p_2=0.52;\ p_3=-0.85$	$\lambda_1=1;\ \lambda_2=\lambda_3=0$	0°
	transverse	3.36	$p_1=0;\ p_2=0.85;\ p_3=0.52$	$\lambda_1=1;\ \lambda_2=\lambda_3=0$	0°
Y	quasi-longitudinal	6.01	$p_1=0;\ p_2=0.91;\ p_3=0.42$	$\lambda_1=0;\ \lambda_2=0.92;\ \lambda_3=0.39$	23°
	transverse	3.92	$p_1=1;\ p_2=p_3=0$	$\lambda_1=0;\ \lambda_2=0.92;\ \lambda_3=-0.39$	−23°
	quasi-transverse	4.35	$p_1=0;\ p_2=0.42;\ p_3=-0.91$	$\lambda_1=0;\ \lambda_2=0.91;\ \lambda_3=-0.42$	−24°
Z	longitudinal	6.32	$p_2=0;\ p_3=1$	$\lambda_1=\lambda_2=0;\ \lambda_3=1$	0°
	transverse	4.68	$p_1=1;\ p_2=p_3=0$	$\lambda_1=0;\ \lambda_2=-0.29,\ \lambda_3=0.96$	−17°
	transverse	4.68	$p_1=p_3=0;\ p_2=1$	$\lambda_1=0;\ \lambda_2=0.29;\ \lambda_3=0.96$	17°

$c_{ik}\cdot 10^{-10}$ dyne/cm². $c_{11} = 87.6$; $c_{33} = 106$; $c_{44} = 58.0$; $c_{66} = 40.7$; $c_{12} = 6.2$; $c_{14} = -17.4$; $c_{13} = 11.9$.

a displacement vector that coincides with the direction of the light beam [6].

The accompanying table lists the calculated angles of deflection of the beams from the normals of waves propagating along the crystallographic axes in quartz. The elastic moduli are found from the measured (using pulse techniques) velocities of sound along the axes;† the value of c_{13} is borrowed from [7]. Deflection of the beams from normal occurs for all three waves propagating along the axis Y, for transverse waves along Z, and not at all along X.

Figures 2-5 are photographs, obtained by the shadow method at 25 Mc, of sound beams in quartz crystal. The sample dimensions were $l^x = 16$ mm, $l_y = l_z = 32$ mm, the direction of observation was aligned with the axis X. Figure 2 is for a longitudinal wave along Z, Fig. 3 for a quasi-longitudinal wave along Y; it is seen that the direction of the reflected and incident beams coincide; Fig. 4 is for simultaneous excitation of a quasi-longitudinal and quasi-transverse wave along Y, Fig. 5 for a quasi-transverse wave reflected from the lateral faces of the sample. Since the wave front is specified by the radiating surface of the crystal, the angles of beam deflection from the normal to the surface in Figs. 3 and 4 correspond to the angles of deflection from the wave normal; their values are very near the analytic.

The distinguishing features of ultrasonic beams propagating in crystals are also brought out by the application of pulse techniques, usually in the study of sound absorption. To obtain a sequence of multiple reflections between opposite faces, the sound beam should not be allowed to be cut short by the sides of the sample. Thus, in quartz, for the case of propagation along Y the sound beams corresponding to transverse and quasi-longitudinal waves are deflected in different directions in the plane YZ. It is therefore possible to observe reflected pulses of each wave only when the radiator-receiver system is situated on a definite portion of the crystal surface. The numbers 1-2 in Fig. 6 outline the interval where only a transverse wave can be "excited," 3-4 only a quasi-longitudinal wave. On the interval 2-3 pulses of both types are detected. Oscillograms appropriate for these cases are shown in Figs. 7a, 7b and 7c (sample dimensions: $l_x = 16$ mm, $l_y = l_z = 32$ mm, $f = 400$ Mc). The excitation and reception

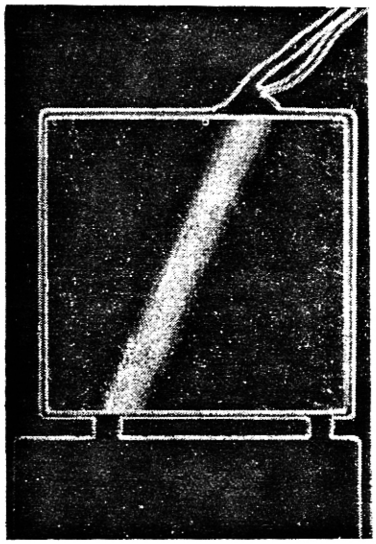

Fig. 3.

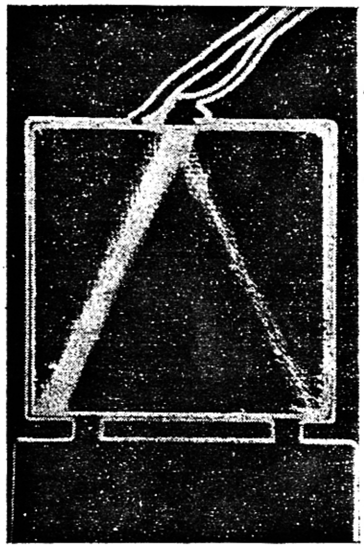

Fig. 4.

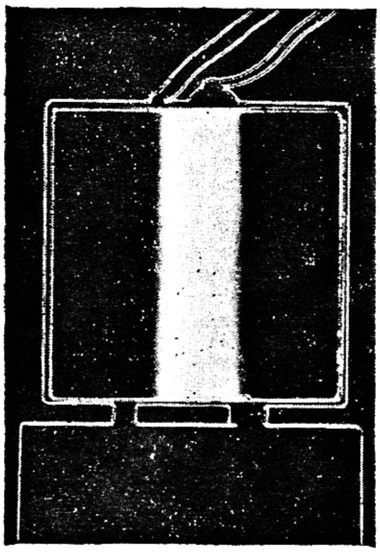

Fig. 2.

† As noted above, the values of c_{ik} thus found correspond to different electric conditions; the elastic moduli listed in the table correspond to $c^D_{11}, c^D_{66}, c^D_{12}$, c^E_{44}, c^E_{14}, $c^E_{13}=c^D_{13}$, $c^E_{33}=c^D_{33}$; because the piezoelectric correction is small this difference was ignored in the calculations.

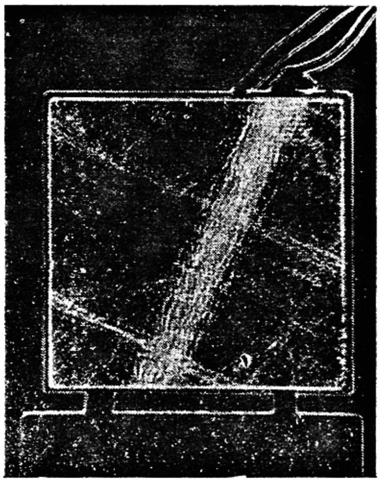

Fig. 5.

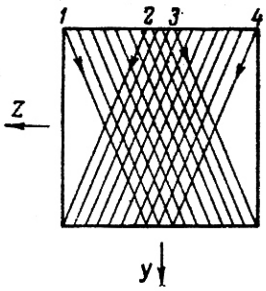

Fig. 6.

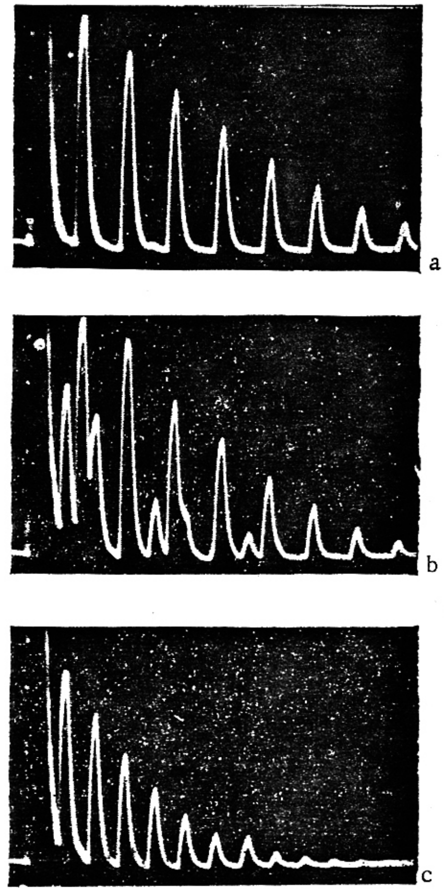

Fig. 7.

of superhigh ultrasonic frequencies were accomplished by a method which we described in [8].

In optically nontransparent crystals the point of emergence of the sound beams on the surface and, consequently, the beam direction are conveniently determined by means of a small rubber damping probe. Applied to the surface at specific points, it aids in observing the reduction in pulse amplitude due to the partial transmission of acoustic energy into the damping probe. We employed this technique for various single crystals at frequencies up to $2 \cdot 10^9$ cps.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.